Multiparameter Harmonic Analysis - Overview

→ We work in $\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2} \otimes \ldots \otimes \mathbb{R}^{n_k}$ (K parameters). What does this mean? Look more closely at the simplest case $\mathbb{R} \otimes \mathbb{R}$: What is the big deal?

→ In harmonic analysis, we are usually concerned w/ operators acting on functions. On $\mathbb{R} \otimes \mathbb{R}$, we look at operators acting on functions $f(x_1, x_2)$, $x_{1,2} \in \mathbb{R}$.

→ Obvious question: How is this so different from working on $\mathbb{R}^2$?

→ Look at the foundations of the 1-parameter theory:

→ Hilbert transform ($\mathbb{R}$):

$K(x) := \frac{1}{x}$

$$\boxed{\begin{aligned} Hf(x) &:= \frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy \\ &= \frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{R}} f(y)K(x-y)\,dy \end{aligned}}$$

→ Riesz transforms ($\mathbb{R}^n$):

$K_j(x) = \frac{x_j}{|x|^{n+1}}$

$$\boxed{\begin{aligned} R_j f(x) &:= c(n)\,\text{p.v.}\int_{\mathbb{R}^n} f(y)\frac{x_j - y_j}{|x-y|^{n+1}}\,dy \\ &= c(n)\,\text{p.v.}\int_{\mathbb{R}^n} f(y)K_j(x-y)\,dy \end{aligned}}$$

→ What happens so differently on $\mathbb{R} \otimes \mathbb{R}$?

$\boxed{f(x_1, x_2)} \longrightarrow \boxed{H_2 f(x_1, x_2)}$ Apply the Hilbert transform in the 2^{nd} variable:

$$H_2 f(x_1, x_2) = \frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{R}} \frac{f(x_1, y_2)}{x_2 - y_2}\,dy_2$$

$\downarrow$

$\boxed{H_1 H_2 f(x_1, x_2)}$ Apply the Hilbert transform in the 1^{st} variable:

$$\begin{aligned} H_1 H_2 f(x_1, x_2) &= \frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{R}} \frac{H_2 f(y_1, x_2)}{x_1 - y_1}\,dy_1 \\ &= \frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{R}} \frac{1}{(x_1 - y_1)}\left(\frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{R}} \frac{f(y_1, y_2)}{x_2 - y_2}\,dy_2\right) \end{aligned}$$

$$\boxed{H_1 H_2 f(x) = \frac{1}{\pi^2}\,\text{p.v.}\int_{\mathbb{R}^2} \frac{f(y_1, y_2)}{(x_1 - y_1)(x_2 - y_2)}\,d(y_1, y_2)}$$ (The "Double Hilbert Transform")

$$= \frac{1}{\pi^2}\,\text{p.v.}\int_{\mathbb{R}^2} f(y_1, y_2)K(x_1 - y_1, x_2 - y_2)\,d(y_1, y_2)$$

$$\boxed{K(x_1, x_2) = \frac{1}{x_1 x_2}}$$

→ Compare to Riesz Kernels on $\mathbb{R}^2$:

$$\boxed{\begin{aligned} K_1(x_1, x_2) &= \frac{x_1}{|x|^3} \\ K_2(x_1, x_2) &= \frac{x_2}{|x|^3} \end{aligned}}$$

→ The fundamental difference: Invariance under dilations.
→ Go back for a moment to the 1-parameter theory:

→ Calderón-Zygmund (convolution) Kernels:

→ The original proof of Marcel Riesz (1928) that the Hilbert transform H preserves $L^p(\mathbb{R})$ used Cauchy's Theorem & complex analysis.
→ The work of Calderón and Zygmund
[1952 - "On the existence of certain singular integrals" - Acta]
introduced real-variable techniques & vastly expanded the theory.
→ They considered convolution operators whose kernel $K(x)$, defined on $\mathbb{R}^n$, "looks like $1/x$ does on $\mathbb{R}^1$"

$$\boxed{Tf(x) := \int_{\mathbb{R}^n} f(y) K(x-y)\,dy}$$

(Later the theory was extended to non-convolution kernels).

→ Important feature: The class of CZ kernels is invariant under dilations

Generally, this means that

$$\boxed{\begin{gathered} x \mapsto \delta x;\ \delta > 0 \\ (x_1, .., x_n) \mapsto (\delta x_1, .., \delta x_n) \end{gathered}}$$

If K is a CZ kernel, then K_δ is also a CZ kernel, where $K_\delta(x) := \frac{1}{\delta^n} K\left(\frac{x}{\delta}\right)$.

Remark: $x \mapsto \delta x$ is a one-parameter family of dilations.

=> the CZ theory seems to be more or less the same independent of the dimension (& # of variables) n.

→ Hilbert Kernel: $K(x) = \frac{1}{x}$; $K_\delta(x) = \frac{1}{\delta} K\left(\frac{x}{\delta}\right) = \frac{1}{\delta} \frac{1}{x/\delta} = \frac{1}{x}$

Riesz Kernels: $K_j(x) = \frac{x_j}{|x|^{n+1}}$; $K_j^\delta(x) = \frac{1}{\delta^n} \frac{x_j/\delta}{|x|^{n+1}/\delta^{n+1}} = \frac{x_j}{|x|^{n+1}}$

→ Double Hilbert Kernel:

$K(x_1, x_2) = \frac{1}{x_1 x_2}$ → Invariant under a two-parameter family of dilations:

$$x = (x_1, x_2) \mapsto (\delta_1 x_1, \delta_2 x_2)$$

$$K_{\delta_1,\delta_2}(x_1, x_2) = \frac{1}{\delta_1\delta_2} K\left(\frac{x_1}{\delta_1}, \frac{x_2}{\delta_2}\right) = \frac{1}{\delta_1\delta_2} \frac{1}{\frac{x_1 x_2}{\delta_1\delta_2}} = \frac{1}{x_1 x_2}$$

What happens if we try this with the Riesz Kernels on $\mathbb{R}^2$?

$$K_j(x) = \frac{x_j}{|x|^{n+1}} = \frac{x_j}{(x_1^2 + x_2^2)^{3/2}} \quad (j = 1, 2)$$

$$\frac{1}{\delta_1\delta_2} K\left(\frac{x_1}{\delta_1}, \frac{x_2}{\delta_2}\right) = \frac{1}{\delta_1\delta_2} \frac{x_j/\delta_j}{\left(\frac{x_1^2}{\delta_1^2} + \frac{x_2^2}{\delta_2^2}\right)^{3/2}}$$

→ Generalization of CZ-theory to the 2-parameter setting $\mathbb{R}^n \otimes \mathbb{R}^m$:
R. Fefferman & Stein: "Singular integrals on product spaces" (Advances 82)

→ They considered "tensor product" operators (like H_1H_2 for example)

$$\mathbb{R}^n \otimes \mathbb{R}^m; \qquad f(\underbrace{x_1, .., x_n}, \underbrace{x_1', .., x_m'})$$

$$Tf = T_{(\mathbb{R}^n)} T_{(\mathbb{R}^m)} f$$

(many proofs are "easier" by appeals to Fubini, Fefferman-Stein inequalities)

→ Journé - "CZO's on Product Spaces" (1985)
Generalized to product space CZO's which are not necessarily of tensor product type - Journé operators.

→ Generally, we are concerned w/ operators acting on functions on

$$\mathbb{R}^{n_1} \otimes \dots \otimes \mathbb{R}^{n_k} \quad (k\text{-parameters})$$

that are invariant under the k-parameter family of dilations

$$\underset{\mathbb{R}^{n_1}}{\underset{\downarrow}{(x_1}}, \dots, \underset{\mathbb{R}^{n_k}}{\underset{\downarrow}{x_k)}} \mapsto (\delta_1 x_1, \dots, \delta_k x_k)$$

→ Side note: There are actually many interesting families of dilations that one can consider. For example: in $\mathbb{R}^3$, we may consider the 2-parameter family $(x_1, x_2, x_3) \mapsto (\delta_1 x_1, \delta_2 x_2, \delta_1\delta_2 x_3)$.

Some defining features of multiparameter harmonic analysis:

→ **The Breakdown of Weak (1,1) inequalities:**

→ Calderón & Zygmund proved that:

$$\|Tf\|_p \le C(n)\,\|f\|_p\,, \quad 1<p<\infty$$

$$|\{x : |Tf(x)| > \lambda\}| \le \frac{C}{\lambda}\|f\|_1\,;\ \lambda>0 \qquad (*)$$

The structure of their proof & techniques developed there set the tone for real-variable theory for many years:

① Prove the L^2-estimate: $\|Tf\|_2 \lesssim \|f\|_2$

② Prove weak (1,1) (CZ decomposition)

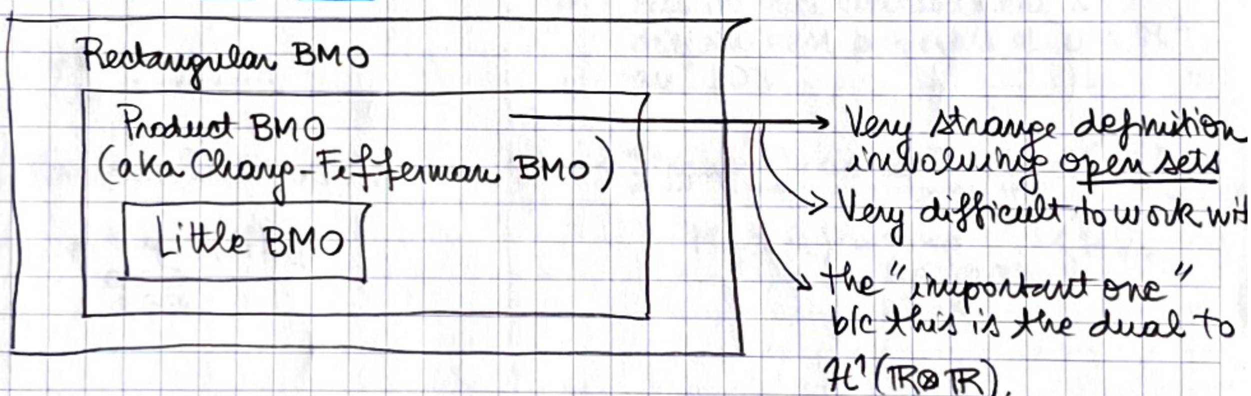
Interpolation: $1<p<2$ ✓

$p>2$: Adjoint ✓

→ When considering classes of kernels invariant under several-parameter dilations, $(*)$ is no longer true

→ Underlying cause: the geometry of rectangles, which leads to (among many consequences) the breakdown of covering lemmas.

→ **There are 3 BMO spaces:** (For example in $\mathbb{R}\otimes\mathbb{R}$)



→ Construction of the Haar basis on $\mathbb{R}^2$:

Recall the Haar functions on $\mathbb{R}$, associated to a dyadic interval I:

$$h_I^0(x) = \frac{1}{\sqrt{|I|}}\left(\mathbb{1}_{I_+} - \mathbb{1}_{I_-}\right) \qquad \text{(cancellative)}$$

$$h_I^1(x) = \frac{1}{\sqrt{|I|}}\,\mathbb{1}_I \qquad \text{(non-cancellative).}$$

Haar Functions on $\mathbb{R}^2$

$$h^{(\varepsilon_1,\varepsilon_2)}_{I_1\times I_2}(x_1,x_2) := h^{\varepsilon_1}_{I_1}(x_1)\,h^{\varepsilon_2}_{I_2}(x)$$

where $|I_1| = |I_2|$

⇒ Four Haar functions:
3 cancellative

$$\begin{array}{|c|c|}\hline + & + \\ \hline - & - \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline - & + \\ \hline - & + \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline - & + \\ \hline + & - \\ \hline\end{array}$$

$(0,1)$ $\quad$ $(1,0)$ $\quad$ $(0,0)$

and 1 non-cancellative

$$\begin{array}{|c|c|}\hline + & + \\ \hline + & + \\ \hline\end{array}$$

$(1,1)$

Dyadic grid: SQUARES
$(\mathcal{D})$

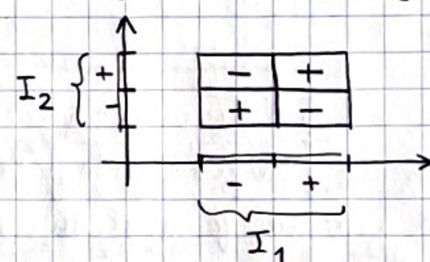
⇒ (dyadic) Maximal function ($\mathbb{R}^n$):

$$Mf(x) := \sup_{\substack{Q\in\mathcal{D}\\ Q\ni x}} \langle |f|\rangle_Q$$

Haar Functions on $\mathbb{R}\otimes\mathbb{R}$

$$h_{I_1\times I_2}(x_1,x_2) := h^0_{I_1}(x_1)\,h^0_{I_2}(x_2) = h^0_{I_1}\otimes h^0_{I_2}$$

No requirement for lengths to be equal!



Dyadic grid $(\mathcal{D}_1\otimes\mathcal{D}_2)$: RECTANGLES

⇒ All the nice containment & disjointness properties of $\mathcal{D}$ in $\mathbb{R}^2$ are LOST.

⇒ Strong Maximal Function ($\mathbb{R}^{\vec{n}}$):

$$M_S f(x) := \sup_{\substack{R\in\mathcal{D}_1\otimes\dots\otimes\mathcal{D}_k\\ R\ni x}} \langle |f|\rangle_R$$

<u>One-parameter BMO:</u> $\|b\|_{BMO_d(\mathbb{R})} := \sup_{I\in\mathcal{D}} \left(\frac{1}{|I|}\int_I |b-\langle b\rangle_I|\right) \simeq \sup_{I\in\mathcal{D}} \left(\frac{1}{|I|}\sum_{K\subset I}(b,h_K)^2\right)^{1/2}$

The second, equivalent, definition comes from John-Nirenberg but also the simple but crucial fact that $\mathbb{1}_I(b-\langle b\rangle_I) = \sum_{K\subset I}(b,h_K)\,h_K$.

Multiparameter BMO spaces ($\mathbb{R}\otimes\mathbb{R}$): Working within $\mathbb{R}\otimes\mathbb{R}$, equipped with dyadic rectangles $\mathcal{R} = \mathcal{D}_1\otimes\mathcal{D}_2$ ($R = I_1\times I_2$; $I_i\in\mathcal{D}_i$; $\mathcal{D}_i$ = dyadic grid on $\mathbb{R}$).

=> Associated Haar system

$$h_R := h_I\otimes h_J,\ \forall R = I\times J\in\mathcal{D}.$$

<u>Rectangular BMO:</u>

$$\|b\|_{BMO_r} := \sup_{R_0\in\mathcal{R}} \left(\frac{1}{|R_0|}\sum_{R\subset R_0}(b,h_R)^2\right)^{1/2}$$

<u>Product BMO:</u>

$$\|b\|_{BMO} := \sup_{\Omega}\left(\frac{1}{|\Omega|}\sum_{\substack{R\in\mathcal{R}\\ R\subset\Omega}}(b,h_R)^2\right)^{1/2}$$

where sup is over all <u>open sets</u> $\Omega\subset\mathbb{R}^2$, $|\Omega|<\infty$.

<u>little bmo:</u>

$$\|b\|_{bmo} := \sup_{R\in\mathcal{R}} \frac{1}{|R|}\int_R |b-\langle b\rangle_R|$$

(iterated) commutators $\big[[b,H_1],H_2\big]$

<u>commutators</u> $[b,T]$, T = Journé operator

$[b,H_1H_2]$

Why do both of the straightforward generalizations fail?
In 2 parameters, there is a disconnect between $\mathbb{1}_R(b-\langle b\rangle_R)$ and $\sum_{T\subset R}(b,h_T)h_T$:

$$\mathbb{1}_R(b-\langle b\rangle_R) = \sum_{T\subset R}(b,h_T)h_T + \sum_{J_1\subset I_1}\left(b, h_{J_1}\otimes\frac{\mathbb{1}_{I_2}}{|I_2|}\right)h_{J_1}(x_1)\mathbb{1}_{I_2}(x_2)$$
$$+ \sum_{J_2\subset I_2}\left(b,\frac{\mathbb{1}_{I_1}}{|I_1|}\otimes h_{J_2}\right)\mathbb{1}_{I_1}(x_1)\,h_{J_2}(x_2)$$

One-parameter formulas: $\langle b\rangle_I = \sum_{J\supsetneq I}(b,h_J)h_J(I)$; $\mathbb{1}_J(b-\langle b\rangle_J) = \sum_{I\subset J}(b,h_I)h_I$

The second formula follows from the first one & properties of Haar functions:

$$\mathbb{1}_J(x)(b(x)-\langle b\rangle_J) = \mathbb{1}_J(x)\Big(\underbrace{\sum_I (b,h_I)h_I(x)\mathbb{1}_J(x)}_{I \text{ must } \cap J} - \sum_{K\supsetneq J}(b,h_K)h_K(J)\mathbb{1}_J(x)\Big)$$

$$= \sum_{I\subset J}(b,h_I)h_I(x) + \sum_{I\supsetneq J}(b,h_I)\cancel{h_I(J)\mathbb{1}_J(x)} - \sum_{K\supsetneq J}(b,h_K)\cancel{h_K(J)\mathbb{1}_J}$$

What happens in 2 parameters? Take $R=I_1\times I_2$. Average formula is the same (basically)

$$\langle b\rangle_R = \frac{1}{|I_1||I_2|}\int_{I_1\times I_2}\Big(\sum_{T\in\mathcal{R}}(b,h_T)h_T(x_1,x_2)\Big)d(x_1,x_2)$$

but $h_T(x_1,x_2)=h_{J_1}(x_1)h_{J_2}(x_2)$ where $T=J_1\times J_2$

$$=\frac{1}{|I_1||I_2|}\sum_{T\in\mathcal R}(b,h_T)\Big(\underbrace{\int_{I_1}h_{J_1}(x_1)}_{0 \text{ unless } I_1\subsetneq J_1}\Big)\Big(\underbrace{\int_{I_2}h_{J_2}(x_2)}_{0\text{ unless } I_2\subsetneq J_2}\Big) = \frac{1}{\cancel{|I_1||I_2|}}\sum_{\substack{J_1\supsetneq I_1\\ J_2\supsetneq I_2}}(b,h_T)h_{J_1}(I_1)\cancel{|I_1|}\,h_{J_2}(I_2)\cancel{|I_2|}$$

$$\boxed{\Rightarrow \langle b\rangle_R = \sum_{\substack{T=J_1\times J_2\\ J_1\supsetneq I_1,\\ J_2\supsetneq I_2}}(b,h_T)\underbrace{h_T(R)}_{h_{J_1}(I_1)h_{J_2}(I_2)} \quad (R=I_1\times I_2)}$$

$\leftarrow$ Remark: this is NOT the same as $\sum_{T\supsetneq R}$ because, for example, $T=J_1\times I_2$ satisfies $T\supsetneq R$.

The other formula takes a turn though:

$$\mathbb 1_R(b-\langle b\rangle_R) = \sum_{T\in\mathcal R}(b,h_T)\underbrace{h_T(x_1,x_2)\mathbb 1_R(x_1,x_2)}_{T=J_1\times J_2} - \mathbb 1_R\langle b\rangle_R$$

$$\Big[\underbrace{h_{J_1}(x_1)\mathbb 1_{I_1}(x_1)}_{\substack{h_{J_1}(x_1) \text{ if } J_1\subseteq I_1\\ h_{J_1}(I_1) \text{ if } J_1\supsetneq I_1}}\Big]\cdot\Big[\underbrace{h_{J_2}(x_2)\mathbb 1_{I_2}(x_2)}_{\substack{h_{J_2}(x_2) \text{ if } J_2\subseteq I_2\\ h_{J_2}(I_2) \text{ if } J_2\supsetneq I_2}}\Big]$$

$$= \sum_{\substack{J_1\subset I_1\\ J_2\subset I_2}}(b,h_T)h_T + \sum_{\substack{J_1\subset I_1\\ J_2\supsetneq I_2}}(b,h_T)h_{J_1}(x_1)h_{J_2}(I_2)\mathbb 1_{I_2}(x_2)$$

$$+\sum_{\substack{J_1\supsetneq I_1\\ J_2\subseteq I_2}}(b,h_T)h_{J_1}(I_1)\mathbb 1_{I_1}(x_1)h_{J_2}(x_2)$$

$$+\cancel{\sum_{\substack{J_1\supsetneq I_1\\ J_2\supsetneq I_2}}(b,h_T)h_T(R)\mathbb 1_R - \langle b\rangle_R\mathbb 1_R}$$

$$=\sum_{\substack{J_1\subset I_1\\ J_2\subset I_2}}(b,h_T)h_T + \sum_{J_1\subset I_1}h_{J_1}(x_1)\Big(\underbrace{\sum_{J_2\supsetneq I_2}(b,h_{J_1}\otimes h_{J_2})h_{J_2}(I_2)}_{(m_{I_2}b,\,h_{J_1})}\Big)\mathbb 1_{I_2}(x_2)$$

$$+\sum_{J_2\subset I_2}h_{J_2}(x_2)\Big(\underbrace{\sum_{J_1\supsetneq I_1}(b,h_{J_1}\times h_{J_2})h_{J_1}(I_1)}_{(m_{I_1}b,\,h_{J_2})}\Big)\mathbb 1_{I_1}(x_1)$$

$$\Big(b,h_{I_1}\otimes\frac{\mathbb 1_{I_2}}{|I_2|}\Big) = \sum_{J_2\supsetneq I_2}(b,h_{I_1\times J_2})h_{J_2}(I_2) = (m_{I_2}b,h_{I_1})_1$$

$$\Big(b,\frac{\mathbb 1_{I_1}}{|I_1|}\otimes h_{I_2}\Big) = \sum_{J_1\supsetneq I_1}(b,h_{J_1\times I_2})h_{J_1}(I_1) = (m_{I_1}b,h_{I_2})_2$$

$$m_{I_1}b(x_2) := \frac{1}{|I_1|}\int_{I_1}b(x_1,x_2)\,dx_1$$

$$m_{I_2}b(x_1) := \frac{1}{|I_2|}\int_{I_2}b(x_1,x_2)\,dx_2$$

$$m_{I_1}b(x_2) = \langle b(\cdot,x_2)\rangle_{I_1} = \sum_{J_1\supsetneq I_1}(b(\cdot,x_2),h_{J_1})_1h_{J_1}(I_1) = \sum_{J_1\supsetneq I_1}\Big(\int b(x_1,x_2)h_{J_1}(x_1)dx_1\Big)h_{J_1}(I_1)$$

$$\Rightarrow (m_{I_1}b,h_{I_2})_2 = \sum_{J_1\supsetneq I_1}\Big(\int b(x_1,x_2)h_{J_1}(x_1)h_{I_2}(x_2)\Big)h_{J_1}(I_1) = \sum_{J_1\supsetneq I_1}(b,h_{J_1}\otimes h_{I_2})h_{J_1}(I_1)$$

$$\mathbb 1_R(b-\langle b\rangle_R) = \sum_{T\subset R}(b,h_T)h_T + \sum_{J_1\subset I_1}(m_{I_2}b,h_{J_1})h_{J_1}(x_1)\otimes\mathbb 1_{I_2}(x_2)$$

$$+ \sum_{J_2\subset I_2}(m_{I_1}b,h_{J_2})\mathbb 1_{I_1}(x_1)\otimes h_{J_2}(x_2)$$

Square function associated with $\{h_R\}_{R\in\mathcal{R}}$: $\quad S_{\mathcal{R}} f(x_1,x_2) := \left(\sum_{R\in\mathcal{R}} (f,h_R)^2 \frac{\mathbb{1}_R}{|R|}\right)^{1/2}$

Dyadic $\mathcal{H}^1$ in this situation: $\|f\|_{\mathcal{H}^1_{\mathcal{R}}(\mathbb{R}\otimes\mathbb{R})} := \|S_{\mathcal{R}} f\|_{L^1(\mathbb{R}\otimes\mathbb{R})}$.

Same construction as before:

$\forall k\in\mathbb{Z}$:
- $U_k := \{x\in\mathbb{R}^2 : S_{\mathcal{R}}\phi(x) > 2^k\}$
- $\tilde{U}_k := \{x\in\mathbb{R}^2 : M_S \mathbb{1}_{U_k}(x) > \frac{1}{2}\}$ Caution here: this is the strong maximal function (which fails weak(1,1) - more later).

$$M_S f(x) = \sup_{\substack{R\in\mathcal{R}\\ R\ni x}} \langle |f|\rangle_R$$

- $\mathcal{R}_k := \left\{R\in\mathcal{R} : |R\cap U_k| > \frac{|R|}{2}\right\}$.

ⓐ $U_{k+1}\subset U_k$; $\mathcal{R}_{k+1}\subset\mathcal{R}_k$ (same)

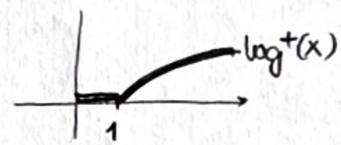
ⓑ $\left(\sum_{k\in\mathbb{Z}} |U_k| 2^k\right) \simeq \|S_{\mathcal{R}}\phi\|_1$ (same)

ⓒ $|\tilde{U}_k| \lesssim |U_k|$ → **slight issue here:** we used weak(1,1) for M here in the 1-param. case

For the strong maximal function:
$$\boxed{|\{x\in\mathbb{R}^2 : M_S f(x) > \lambda\}| \lesssim \int_{\mathbb{R}^2} \frac{|f(x)|}{\lambda}\left(1+\log^+\frac{|f(x)|}{\lambda}\right)dx}$$

Still, this is OK for us:

$$|\tilde{U}_k| = |\{M_S\mathbb{1}_{U_k} > \tfrac{1}{2}\}|$$
$$\lesssim \int_{\mathbb{R}^2} \frac{\mathbb{1}_{U_k}}{1/2}\left(1+\log^+\frac{\mathbb{1}_{U_k}}{1/2}\right)dx$$
$$= \int_{U_k} 2\left(1+\log^+(2)\right)dx \simeq |U_k|.$$



ⓓ $\left(\bigcup_{R\in\mathcal{R}_k} R\right)\subset\tilde{U}_k$ (same)

ⓔ $R\notin\bigcup_{k\in\mathbb{Z}}\mathcal{R}_k \Rightarrow |R\cap\{S_{\mathcal{R}}=0\}| \geq \frac{|R|}{2} \Rightarrow (\phi,h_R)=0$ (same)

ⓕ $\bigcap_{k\in\mathbb{Z}}\mathcal{R}_k = \phi$ (same)

$$|(b,\phi)| \leq \sum_{R\in\mathcal{R}} |(b,h_R)||(\phi,h_R)| = \sum_{k\in\mathbb{Z}} \underbrace{\left(\sum_{R\in\mathcal{R}_k\setminus\mathcal{R}_{k+1}} (b,h_R)^2\right)^{1/2}}_{\lesssim \|b\|_{BMO}|\tilde{U}_k|^{1/2}} \underbrace{\left(\sum_{R\in\mathcal{R}_k\setminus\mathcal{R}_{k+1}} (\phi,h_R)^2\right)^{1/2}}_{\lesssim 2^k|\tilde{U}_k|^{1/2} \text{ (same)}}$$

$$\left(\sum_{R\in\mathcal{R}_k\setminus\mathcal{R}_{k+1}} (b,h_R)^2\right)^{1/2} \leq \left(\sum_{R\in\mathcal{R}_k} (b,h_R)^2\right)^{1/2}$$
$$\underset{ⓓ}{\leq} \left(\sum_{\substack{R\in\mathcal{R}\\ R\subset\tilde{U}_k}} (b,h_R)^2\right)^{1/2}$$
$$\leq |\tilde{U}_k|^{1/2}\|b\|_{BMO_{\mathcal{R}}(\mathbb{R}\otimes\mathbb{R})} \quad ! \text{ (easier even)}$$